# On Quantum-Classical Hybrid Canonical Dynamics

Lajos Diósi

Wigner Research Center for Physics

Eötvös Loránd University

*

Supports by: FQXi (Grant No. FQXi-RFP-CPW-2008), National Research, Development and Innovation Office for "Frontline" Research Excellence Program (Grant No. KKP133827) and John Templeton Foundation (Grant 62099).

21 June 2023, Guildford

# Abstract

Quantum-classical hybrid dynamics cannot retain the reversibility of the constituent quantum and classical dynamics. If, for instance, the classical constituent is canonical and we construct a deterministic hybrid dynamics using the sum of the Dirac and Poisson brackets, the positivity of the hybrid density is not preserved. For a legitimate Markovian dynamics, we should impose additional decoherence and diffusion mechanisms respectively on the quantum and classical evolutions. It turns out that the product of the decoherence and diffusion coefficients cannot be smaller than the strength of hybrid coupling. This implies a condition for the minimum of the mandatory irreversibility of hybrid dynamics.

# Classical, Quantum, Hybrid

Desires of hybrid dynamics

| | CLASSICAL subSYSTEM | QUANTUM subSYSTEM |
|---|---|---|
| chemistry | nuclei | electrons |
| cosmology | gravity | matter |
| foundations | measuring device | measured system |
| open systems | reservoir | system of interest |
| control | measured signal | controlled system |

# Coupling Classical and Quantum Formalisms

| | C subSYSTEM | Q subSYSTEM |
|---|---|---|
| State: | $\rho(q,p)\geq 0$ | $\hat\rho\geq 0$ |
| Hamilton: | $H(q,p)$ | $\hat H$ |
| Motion: | $\dot\rho=\{H,\rho\}$ | $\dot{\hat\rho}=-\frac{i}{\hbar}[\hat H,\hat\rho]$ |
| | Liouville eq. | von Neumann eq. |
| | Poisson br. | Dirac br. |
| | ⇒coupling⇐ | |
| State: | $\hat\rho(q,p)\equiv\widehat{\rho}\geq 0$ | |
| Hamilton: | $\hat H(q,p)\equiv\widehat{H}$ | |
| Motion: | $\dot{\widehat{\rho}}=-\frac{i}{\hbar}[\widehat{H},\widehat{\rho}]+\mathrm{Herm}\{\widehat{H},\widehat{\rho}\}$ | |
| | Aleksandrov–Gerasimenko hybrid eq. | |
| | R.H.S.: Aleksandrov bracket | |

# AG Hybrid Dynamics 1981/82

$$\dot{\widehat{\rho}} = -\frac{i}{\hbar}[\widehat{H}, \widehat{\rho}] + \mathrm{Herm}\{\widehat{H}, \widehat{\rho}\}$$

$$\begin{aligned} \dot{\hat{\rho}}(q,p) = & - \frac{i}{\hbar}\left[\hat{H}(q,p), \hat{\rho}(q,p)\right] + \\ & + \mathrm{Herm}\left(\frac{\partial \hat{H}(q,p)}{\partial p}\frac{\partial \hat{\rho}(q,p)}{\partial q} - \frac{\partial \hat{H}(q,p)}{\partial q}\frac{\partial \hat{\rho}(q,p)}{\partial p}\right) \end{aligned}$$

Useful effective dynamics. But inconsistent mathematically.
D.-Gisin-Strunz 2000: 1D classical particle coupled to Pauli-spin:

$$\hat{H}(q,p) = \hat{H}_Q + (p^2/2m) + \kappa p \hat{\sigma}_3$$

AG hybrid eq. can destroy positivity $0 \leq \hat{\rho}(q,p)$.
So what?

# All Quantum Dynamics

General form of legitimate quantum dynamics (with Einstein's summation rule)

$$\dot{\hat{\rho}} = -\frac{i}{\hbar}[\hat{H}, \hat{\rho}] + \left(\hat{L}_\alpha \hat{\rho} \hat{L}_\alpha^\dagger - \mathrm{Herm} \hat{L}_\alpha^\dagger \hat{L}_\alpha \hat{\rho}\right)$$

where operators $\hat{I}, \hat{L}_1, \ldots, \hat{L}_\alpha, ...$ form a linearly independent system. If this r.h.s. structure does not exists, positivity $\hat{\rho} \geq 0$ will be violated when our system is suitably entangled with an idle quantum system (sometimes in the absence of such entanglement as well).

# Quantum Dynamics with Two $\hbar$'s, D. 1995

- Quantize the classical subsystem as well, but with $\hbar' \neq \hbar$
- Couple it to the quantum subsystem of interest. Educated Ansatz: generalization of Dirac bracket $-(i/\hbar)[.,.]$ for two $\hbar$'s.
- Nonunitary dynamics, like a quantum master eq. But!
- Incomplete Lindblad 1976 (GKLS 1976, in fact) master eq.
- Complete it! Add the minimum necessary new terms.
- Take $\hbar' \to 0$

# Positivity Preserving Hybrid Dynamics, D. 1995

$$\begin{aligned} \hat{H}(q,p) &= \hat{H}_Q + H_C(q,p) + C(q,p)\hat{Q} \\ \widehat{H} &= \hat{H}_Q + H_C + C\hat{Q} \end{aligned}$$

$$\dot{\widehat{\rho}} = -\frac{i}{\hbar}[\widehat{H},\widehat{\rho}] + \mathrm{Herm}\{\widehat{H},\widehat{\rho}\} - \frac{\lambda}{4\hbar^2}[\hat{Q},[\hat{Q},\widehat{\rho}]] + \frac{1}{4\lambda}\{C,\{C,\widehat{\rho}\}\}$$

AG dynamics — decoherence — diffusion

Least added noise: (stregth of decoh.) $\times$ (strength of diff.) $=$ const.

Example: 1D classical particle coupled to Pauli-spin:

$$\hat{H}(q,p) = \hat{H}_Q + (p^2/2m) + \kappa p \hat{\sigma}_3$$

$$\begin{aligned} \dot{\hat{\rho}}(q,p) &= -\frac{i}{\hbar}[\hat{H}_Q,\hat{\rho}(q,p)] + \frac{p}{m}\frac{\partial\hat{\rho}(q,p)}{\partial q} + \kappa \mathrm{Herm}\hat{\sigma}_3\hat{\rho}(q,p) \\ &\quad -\frac{\lambda\kappa^2}{4\hbar^2}[\hat{\sigma}_3,[\hat{\sigma}_3,\hat{\rho}(q,p)]] + \frac{\kappa^2}{4\lambda}\frac{\partial^2\hat{\rho}(q,p)}{\partial q^2} \end{aligned}$$

Positivity $0 \le \hat{\rho}(q,p)$ guaranteed.

# All Completely Positive Hybrid Dynamics

Oppenheim et al. 2022, D. 2023

Generic form of coupling continuous classical variables $x$ to quantum dynamics:

$$\frac{d\hat{\rho}}{dt} = -i[\hat{H},\hat{\rho}] + \frac{\partial}{\partial x^n}(\overline{G}^{n\alpha}_{CQ}\hat{L}_\alpha\hat{\rho} + H.C.)$$

$$+D_Q^{\beta\alpha}\left(\hat{L}_\alpha\hat{\rho}\hat{L}_\beta^\dagger - \mathrm{Herm}\hat{L}_\beta^\dagger\hat{L}_\alpha\hat{\rho}\right) + \frac{1}{2}\frac{\partial^2}{\partial x^n \partial x^m}(D_C^{nm}\hat{\rho}) - \frac{\partial}{\partial x^n}(V^n\hat{\rho})$$

Every object is function of $x$, including the matrix coeffecients $G_{CQ}$ (backaction), $D_Q, D_C$ (decoherence, diffusion).

If $G_{CQ} \neq 0$, noise is mandatory. If, e.g., $G_{CQ} = gI$:

$$D_Q D_C \geq g^2$$

Canonical dynamics: coordinates $\{x^n; n = 1, ..., N\}$, momenta $\{x^n; n = N+1, ..., 2N\}$, hybrid coupling $\hat{H}_{CQ}(x) = h^\alpha(x)\hat{L}_\alpha$:

$$G^{n\alpha}_{CQ}(x) = -\tfrac{1}{2}\epsilon^{nm}\frac{\partial h^\alpha(x)}{\partial x^n}$$

# Summary, Outlook

Summary:

- Hybrid coupling invokes irreversibility (noise)
- Aleksandrov bracket must be completed by decoherence and diffusion
- Generic hybrid dynamics contain exact lower limit on noise

Outlook (results not discussed here)

- Hybrid master equations can be unravelled into random trajectories
- ... helpful for MC simulations on one hand
- ... equivalent to standard quantum monitoring on the other
- ... and equivalent with formalisms of sponaneous collapse theories
- Top motivation in foundations: if gravity remains classical?

# References

IV Aleksandrov 1981 Z. Naturforschung **36a**. 902
V Gerasimenko 1982 Theor. Math, Phys. **50**, 49
R Kapral 2006 Ann. Rev. Phys. Chem. **57**, 129
L Diósi, N Gisin, WT Strunz 2000 Phys.Rev. **A61**, 22108
L Diósi 1995 Quantum dynamics with two Planck constants and the semiclassical limit quant-ph/9503023
L Diósi 2014 Phys. Scr. **T163** 014004
J Oppenheim et al. 2022 The two classes of hybrid classical-quantum dynamics arXiv:2203.01332
L. Diósi 2023 Phys.Rev. **A107**, 062206